Working Model Technical Note: Modeling Uniform Flexible Bodies in Working Model

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DOC #:

Summary

Although bodies in a Working Model simulation are considered rigid, Working Model can analyze the motion of flexible bodies. This paper describes the steps necessary to perform accurate dynamic simulations of systems that include flexible bodies.

Introduction

Realistically, no body is perfectly rigid. When a load is applied to a body, that body deforms. However, Working Model models bodies as perfectly rigid for two reasons:

1. **Most real-world objects are stiff enough such that the results of rigid-body dynamics are very accurate.**

2. **Flexible-body dynamics is very computationally intensive.**

Since the dynamics of slightly flexible bodies are often dominated by the rigid-body motion, one does not lose much accuracy by neglecting the small deformations.

However, the rigid-body assumption is less accurate with objects that are more flexible. To accurately simulate flexible-body motion, analysts often use finite-element analysis programs.

Finite Element Analysis

Engineers sometime have the misconception that finite-element programs are the perfect way to simulate the motion of a flexible body. Finite element programs, as powerful as they are, have drawbacks.

Most finite element programs are linear codes that assume small displacements. These programs cannot simulate the motion of flexible bodies which are moving (i.e., undergoing translation or rotation). The term "linear" means the motion is governed by linear differential equations. This is not to be confused with linear stress-strain relationship, as most linear codes can handle nonlinear stress-strain relationships.

Figure 1
Two systems with flexible elements

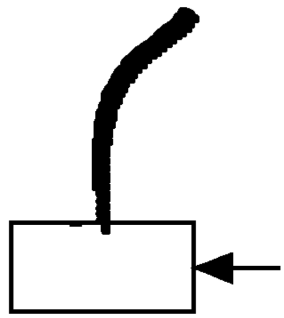


Figure 1a
Linear Motion

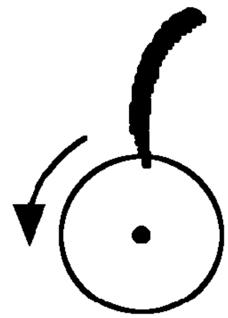


Figure 1b
Rotational Motion

For example, note the systems pictured in Figure 1. Both systems have a flexible beam attached to a rigid body. The system in Figure 1a undergoes linear motion (governed by linear differential equations) and can be modeled by most finite element packages. However, the system in Figure 1b undergoes rotational motion (which is governed by nonlinear differential equations) and cannot be modeled by most finite element packages.

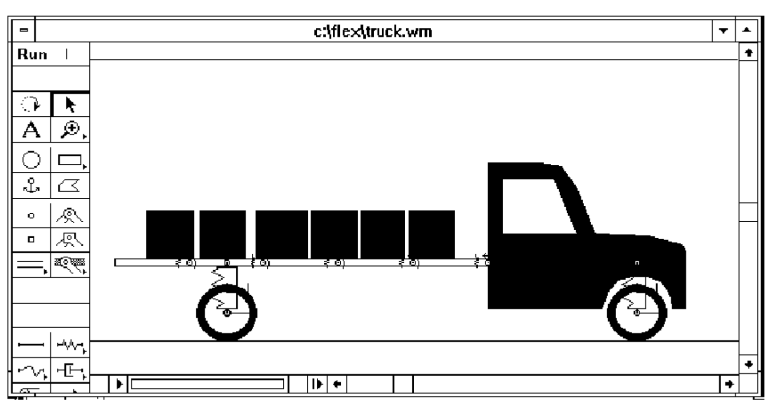
There are a few finite element packages (called nonlinear codes) that can simulate the motion of the system in Figure 1b. However these codes are more costly and slower than the conventional linear finite element programs.

Furthermore, finite element codes have the limitation than every body in the simulation must be analyzed as a flexible body. Large simulation times result from systems with a mixture of rigid and flexible bodies. Rigid bodies possess very high flexible body frequencies, slowing down the program's numerical integration.

Flexible Bodies which can be analyzed by Working Model

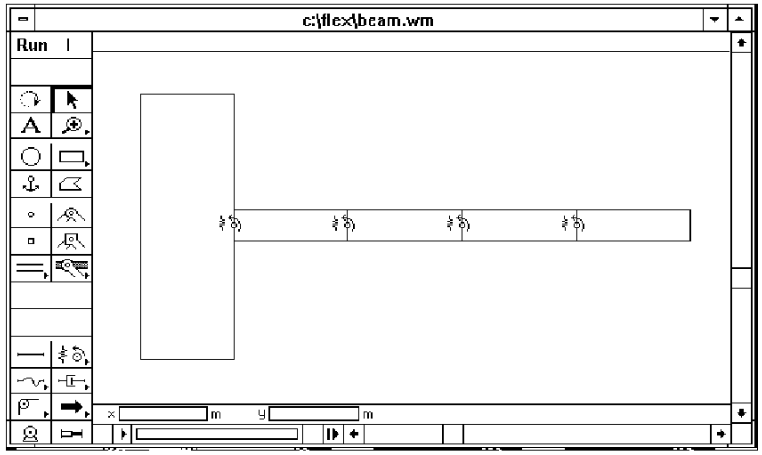
In general, Working Model cannot model flexible bodies. However, Working Model can accurately simulate the motion of flexible beams. Thus, if an object can be modeled as a beam, its motion can be accurately simulated by Working Model. For example, since the flexible bed of the truck pictured in Figure 2 can be modeled as a beam, its motion can be accurately simulated by Working Model.

Figure 2
Truck with Flexible Bed



In Working Model, beams are modeled by breaking them down into discrete sections (of equal size) which are connected by rotational springs. Although each section is still a rigid body, the rotational springs allow the beam to flex as the actual flexible body does. Figure 3 shows such a beam that has been broken into discrete sections.

Figure 3
Discrete Sectioned Flexible Beam Connected to a Rigid Body



Assigning Spring Constants

When a flexible body is discrete sectioned (as in Figure 3), rotational springs are introduced to connect the segments. The spring constants of these springs must be chosen carefully in order for the simulation to produce results that accurately match the "real world" results.

For springs located at cantilevered interface, the spring constant is:

$$K = \frac{E*I}{L}$$

For springs located between segments, the spring constant is:

$$K = \frac{E*I}{L} * \frac{6*N}{3*N-1}$$

Where:

E = Elastic Modulus of the Flexible Material
I = Area Moment of Inertia about the Bending Axis
N = Number of Discrete Sectioned Elements
L = Length of each Discrete Sectioned Element (Total Length = L*N)

Elastic Modulus Examples

The Elastic Modulus describes the flexibility of the material. A material with a large Elastic Modulus is stiff, while a material with a small Elastic Modulus is flexible. The Elastic Modulus of some common materials are:

Wood	1-2 million lbf/in^2
Concrete	3 million lbf/in^2
Marble	8 million lbf/in^2
Aluminum	10 million lbf/in^2
Brass	15 million lbf/in^2
Cast Iron	10-20 million lbf/in^2
Steel	30 million lbf/in^2

Area Moment of Inertia

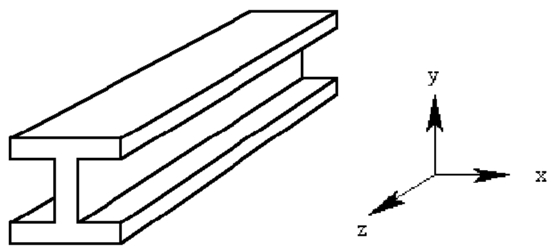
Note that I is the central area moment of inertia of the beam's cross-section. The area moment of inertia (which is widely used in classical beam theory) should not be confused with the mass moment of inertia (which is commonly used in rigid-body dynamics).

Although Figure 4 shows displays a typical I-beam, Working Model can analyze the motion of beams with any uniform cross-section. The area moment of inertia is calculated by integrating the following formula over the beam's cross-section:

$$\iint d^2 \, dA$$

Where d is the distance from the dA point to a line that is parallel to the vector **x** and passes through the centroid of the cross-section.

Figure 4
Typical Beam and Coordinate System



Example: Cantilever/Free

The system pictured in Figure 3 consists of a flexible beam rigidly attached (cantilevered) to a rigid body. The flexible beam is discretized into 4 elements. The leftmost spring is assigned a spring constant:

$$K = \frac{E*I}{L} * \frac{24}{11}$$

while the other three springs are assigned spring constants:

$$K = \frac{E*I}{L}$$

Example: Cantilever/Cantilever

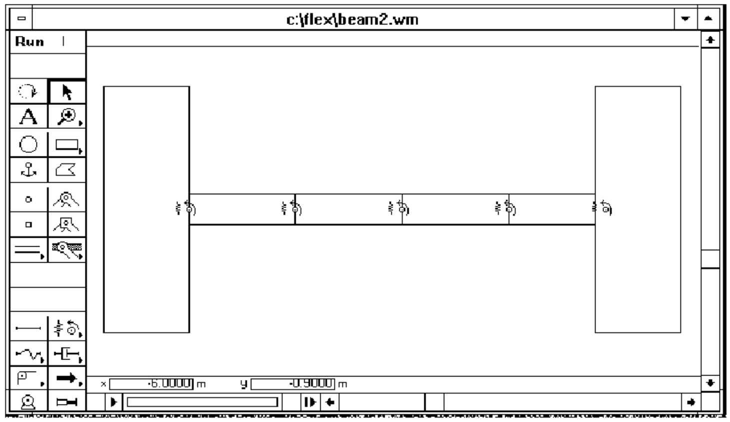
The system pictured in Figure 5 consists of a flexible beam rigidly attached (cantilevered) to two rigid bodies. The flexible beam is discretized into 4 elements.
The leftmost spring and rightmost springs are assigned spring constants:

$$K = \frac{E*I}{L} * \frac{24}{11}$$

while the other three springs are assigned spring constants:

$$K = \frac{E*I}{L}$$

Figure 5
Discrete Sectioned Flexible Beam Connected to Two Rigid Bodies



Choosing the Number of Discrete Sectioned Elements

The above section outlines how to choose spring constants when a flexible beam is discrete sectioned into N equally sized sections. This section describes how to choose N.

As the number of sections is increased, the accuracy improves. Since execution time greatly slows with more elements, the user should not choose more elements than necessary to meet the desired error criteria. Figures 6 and 7 show the error in the first three natural frequencies as a function of the number of elements.

For a relatively rigid beam, the motion of the beam will be dominated by the first mode. Thus, the user could choose the number of elements by inspection of the "First Frequency" curve in Figure 6 or 7. However, for more flexible beams, the contribution of the second and third modes become significant. In those cases, the user must choose the number of elements by inspection of the other curves in Figure 6 or 7.

There is no hard-and-fast rule for how many elements are needed. The graphs in Figures 6 and 7 give a good starting point, but the only way to verify that your discretization is accurate is to re-run the simulation with more elements. If the results of the two simulations agree, you have good results.

Figure 6
Error vs. Elements for Cantilever/Free Beam

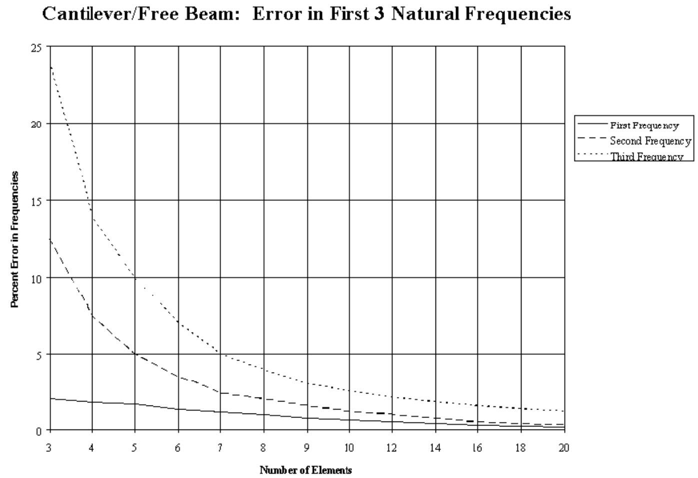
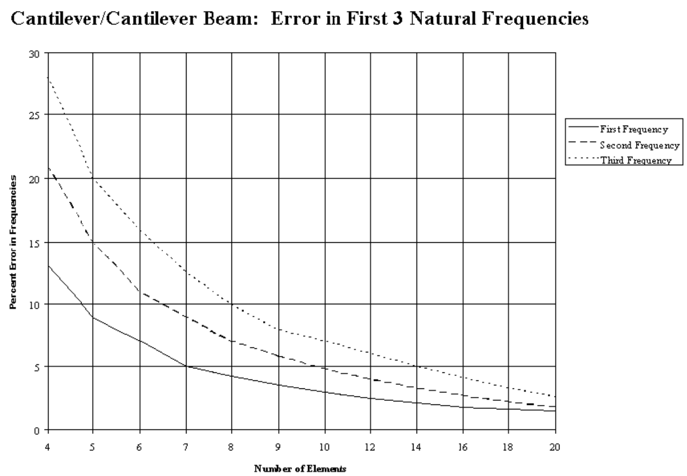


Figure 7
Error vs. Elements for Cantilever/Cantilever Beam



Importance of Integration Time Step

The motion of flexible bodies involves frequencies that are much higher than those encountered in rigid-body motion. Since these high frequencies are hard to numerically integrate, the integrator time step must be decreased. The size of the integrator time step depends on how many modes you expect to simulate accurately. The integration time step is set by following procedure:

1. From Figures 6 and 7, determine how many modes you expect to accurately simulate. Use the number of modes to determine the constant C from the table below:

Number of Modes	Constant K
1	0.357
2	0.130
3	0.066
4	0.040
5	0.027

2. Calculate the maximum step size h by the formula:

$$h = C*L^2*N^2 * \sqrt{\frac{p*A}{E*I}}$$

   Where:
   C is from step #1
   L = Length of each Discrete Sectioned Element
   N = Number of Discrete Sectioned Elements
   p = Density of Material (mass per unit volume)
   A = Cross-sectional Area of Beam
   E = Elastic Modulus of the Flexible Material
   I = Area Moment of Inertia about the Bending Axis

3. Choose *Accuracy* from the *World* menu
4. Click on Custom
5. Click on Runge Kutta 4 Integrator
6. Click on variable time step
7. Enter h (calculated in step #2) in the Animation Step box.

These steps result in slow execution, but they are necessary to ensure that the flexible-body motion is accurately simulated.

Derivation Overview

Most of this research was performed by Paul Mitiguy in conjunction with projects at NASA Ames Research Center.

The spring constant values described in the "Discretization Rules" section, were derived by comparing the static results of the discretized beam with Euler beam theory. The values for the spring constants were backed out by matching the deflections of the joints of the discretized beam with those predicted by Euler beam theory. The equations for Euler beam theory are found in many reference books including *Formulas for Stress and Strain* by Roark and Young and *Mechanical Engineering Design* by Shigley and Mitchell.

In order to produce the graphs in Figures 6 and 7, the full, nonlinear equations of motion for the discretized beam were formed. Then the equations were linearized and the frequencies were extracted by eigen analysis. These frequencies were compared to the actual frequencies found in many reference books including *Formulas for Stress and Strain* by Roark and Young, pages 576-579.

The equation for the step size h (in step #2 of the previous section) was found by examining the first three natural frequencies of a uniform beam with both ends fixed (pp. 576 of *Formulas for Stress and Strain* by Roark and Young). These frequencies were then used to determine the maximum Runge Kutta steps size necessary for stable integration (pp. 41 of *Numerical Initial Value Problems in Ordinary Differential Equations* by C. William Gear).

This section is not meant as a full description of how these discretization rules were derived. For more information, contact Knowledge Revolution.

Appendix: An Introduction to Modal Analysis

The dynamics of a flexible body is often analyzed by a method called *Modal Analysis*. Let y(x,t) represent the deflection of a point at time t at a distance x along a beam. The solution to y(x,t) is governed by PDEs (partial differential equations). Since PDEs are hard to solve, it is often advantageous to separate the problem such that it is governed by sets of ODEs (ordinary differential equations). The separation can often be achieved by assuming y(x,t) is an infinite summation:

$$y(x,t) = p_1(x)*q_1(t) + p_2(x)*q_2(t) + p_3(x)*q_3(t) + \ldots$$

Where the $p_k(x)$ terms are called *Mode Shapes* and the $q_k(t)$ terms are called *Modal Coefficients*. The terms are ordered such that $q_1(t)$ has the lowest frequency, $q_2(t)$ has the next lowest frequency, etc. Although this series contains an infinite number of terms, the Modal Coefficients $q_k(t)$ gets smaller as k gets larger. As a result, the motion can often be accurately approximated by only a few modes.

Example: Motion of a Pinned-Pinned Beam

The first three flexible modes for a beam pinned at each end are determined to be:

$$p_1(x) = \sin(\pi x/L)$$
$$p_2(x) = \sin(2\pi x/L)$$
$$p_3(x) = \sin(3\pi x/L)$$

where x is the distance from the left end of the beam and L is the total length of the beam. Below are sketches of the first three mode shapes:

Figure A1 First Mode Shape $p_1(x)$

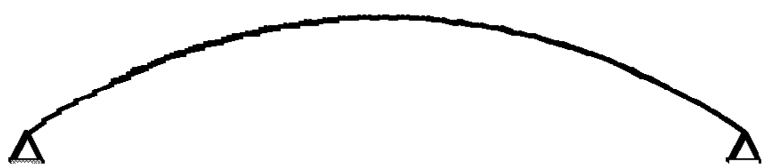


Figure A2
Second Mode Shape
$p_2(x)$

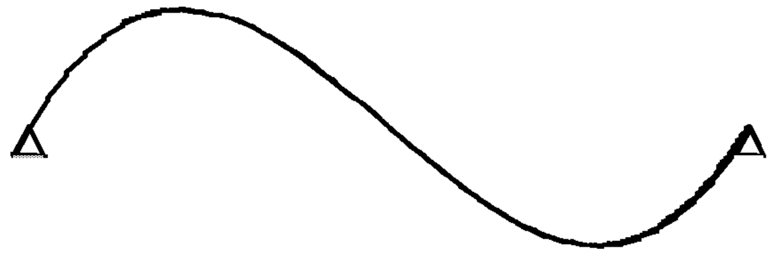
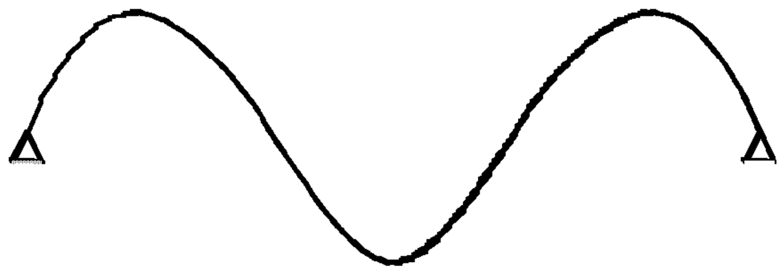


Figure A3
Third Mode Shape
$p_3(x)$



Once the time-histories of $q_1(t)$, $q_2(t)$, and $q_3(t)$ are known, the deflection of the beam at time t at a distance x along the beam can be approximated by truncating the infinite series after three terms:

$$y(x,t) = p_1(x)*q_1(t) + p_2(x)*q_2(t) + p_3(x)*q_3(t)$$

Unfortunately, there is no way to tell how many modes must be kept for accurate analysis. The only sure way is to perform the analysis with a certain number of modes and then perform the analysis again with additional mode(s). If the results of the two analyses agree, the results are probably accurate.